曲线坐标网格下数值计算技术注记

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摘要:在守恒量形式上保持守恒的研究工作基础上,进一步给出了让守恒律型方程在形式上保持守恒律形式的一般性处理技术.对于直接涉及到计算能否进行的可行性问题,给出了把物理域中的不渗透固壁边界条件经过曲线坐标网格生成变换后转成在计算域中形式的技术.

关键词:守恒律;曲线坐标网格;边界条件

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在高维(维数大于 1)流场的数值模拟中,由于边界一般不规则,通常采取曲线坐标网格方法或无结构网格方法.曲线坐标网格方法就是通过把不规则的物理域映射到规则的计算域,同时控制方程也进行变换,然后在规则的计算域上进行规则的网格剖分和控制方程离散.对于通常的守恒律型方程,存在这样两个问题:(a)原来的守恒律型方程还能不能在形式上保持守恒律形式;(b)原来的边界条件在计算域上该如何给出.对于前一个问题,如果在计算域中能在形式上保持守恒律形式,则可以把一些守恒型的差分格式和差分格式的设计思想直接移植到曲线坐标系下的控制方程离散化上,这对于设计满足诸如守恒型性质的计算格式来讲是重要的.有不少文献针对某些较具体形式的方程给出了变换后的情况[1],本文在前面的守恒量形式上保持守恒的工作基础上[2],通过举例给出了守恒律型方程在形式上保持守恒律形式的一般性处理技术.后一个问题直接涉及到计算能否进行的可行性,有不少文献追求曲线坐标满足正交性,其实根本原因就是想消除或简化边界条件处理的复杂性.但生成满足正交性的曲线坐标网格是一件较困难的事情,通常得到的都是近似满足正交性的曲线坐标网格,因此有必要探讨在一般的曲线坐标网格生成变换下,物理域上的边界条件在计算域上如何给出的问题.这里为了在计算域中数值求解的需要,给出了把物理域中的不渗透固壁边界条件经过曲线坐标网格生成变换后转成在计算域中形式的技术.

1 守恒型方程在形式上保持守恒形式的技术

根据文献[2]知道,如果假设

$$\begin{cases} x_1 = x_1(\eta_1,\cdots,\eta_n) \\ x_2 = x_2(\eta_1,\cdots,\eta_n) \\ \qquad \vdots \\ x_n = x_n(\eta_1,\cdots,\eta_n) \end{cases} \tag{1}$$

是把计算域 D_c 一一映上到物理域 D 的具有二阶连续偏导数的映射,且相应的逆映射

$$\begin{cases} \eta_1 = \eta_1(x_1,\cdots,x_n) \\ \eta_2 = \eta_2(x_1,\cdots,x_n) \\ \qquad \vdots \\ \eta_n = \eta_n(x_1,\cdots,x_n) \end{cases} \tag{2}$$

具有二阶连续偏导数,则有

$$\sum_{j=1}^{n} \frac{\partial}{\partial \eta_j}\left(J \frac{\partial \eta_j}{\partial x_i}\right) = 0 \tag{3}$$

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成立.其中

$$J = \left|\frac{\partial(x_1,\cdots,x_n)}{\partial(\eta_1,\cdots,\eta_n)}\right|$$

$$\frac{\partial(x_1,\cdots,x_n)}{\partial(\eta_1,\cdots,\eta_n)} = \begin{pmatrix} \frac{\partial x_1}{\partial \eta_1} & \cdots & \frac{\partial x_1}{\partial \eta_n} \\ \square & \square & \square \\ \frac{\partial x_n}{\partial \eta_1} & \cdots & \frac{\partial x_n}{\partial \eta_n} \end{pmatrix}$$

如果 $W = W(x_1, x_2, \cdots, x_n)$ 或写成 $W = W(\eta_1, \eta_2, \cdots, \eta_n)$,是具有一阶导数的守恒物理量,则有

$$JW_{x_i} = J\sum_{j=1}^{n} W_{\eta_j}\frac{\partial \eta_j}{\partial x_i}$$

根据式(3) 应有

$$JW_{x_i} = \sum_{j=1}^{n}\left[W_{\eta_j}\left(J\frac{\partial \eta_j}{\partial x_i}\right) + W\frac{\partial}{\partial \eta_j}\left(J\frac{\partial \eta_j}{\partial x_i}\right)\right]$$

从而得到重要关系式

$$JW_{x_i} = \sum_{j=1}^{n}\frac{\partial}{\partial \eta_j}\left(J\frac{\partial \eta_j}{\partial x_i}W\right) \tag{4}$$

下面通过举例给出把守恒律型方程在形式上还变成守恒律型方程的一般性处理技术.现假设

$$\tau = t \qquad \eta_1 = \eta_1(t,x,y) \qquad \eta_2 = \eta_2(t,x,y) \tag{5}$$

是把二维笛卡尔坐标系下的物理域 D 一一映上到计算域 D_c 的具有二阶连续导数的映射.记

$$J = \left|\frac{\partial(t,x,y)}{\partial(\tau,\eta_1,\eta_2)}\right| = \begin{vmatrix} 1 & 0 & 0 \\ \frac{\partial x}{\partial \tau} & \frac{\partial x}{\partial \eta_1} & \frac{\partial x}{\partial \eta_2} \\ \frac{\partial y}{\partial \tau} & \frac{\partial y}{\partial \eta_1} & \frac{\partial y}{\partial \eta_2} \end{vmatrix} \tag{6}$$

设二维直角坐标系下的守恒型方程形式为

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial x} + \frac{\partial \boldsymbol{G}(\boldsymbol{U})}{\partial y} = \boldsymbol{0} \tag{7}$$

据式(4) 知

$$J\frac{\partial \boldsymbol{U}}{\partial t} = \frac{\partial}{\partial \tau}(J\boldsymbol{U}) + \frac{\partial}{\partial \eta_1}\left(J\frac{\partial \eta_1}{\partial t}\boldsymbol{U}\right) + \frac{\partial}{\partial \eta_2}\left(J\frac{\partial \eta_2}{\partial t}\boldsymbol{U}\right)$$

$$J\frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial x} = \frac{\partial}{\partial \eta_1}\left(J\frac{\partial \eta_1}{\partial x}\boldsymbol{F}(\boldsymbol{U})\right) + \frac{\partial}{\partial \eta_2}\left(J\frac{\partial \eta_2}{\partial x}\boldsymbol{F}(\boldsymbol{U})\right)$$

$$J\frac{\partial \boldsymbol{G}(\boldsymbol{U})}{\partial y} = \frac{\partial}{\partial \eta_1}\left(J\frac{\partial \eta_1}{\partial y}\boldsymbol{G}(\boldsymbol{U})\right) + \frac{\partial}{\partial \eta_2}\left(J\frac{\partial \eta_2}{\partial y}\boldsymbol{G}(\boldsymbol{U})\right)$$

式(7)两边同时乘以 J,并利用上面的关系式可以得到

$$\frac{\partial}{\partial \tau}(J\boldsymbol{U}) + \frac{\partial}{\partial \eta_1}\left(J\frac{\partial \eta_1}{\partial t}\boldsymbol{U}\right) + \frac{\partial}{\partial \eta_2}\left(J\frac{\partial \eta_2}{\partial t}\boldsymbol{U}\right) + \frac{\partial}{\partial \eta_1}\left(J\frac{\partial \eta_1}{\partial x}\boldsymbol{F}(\boldsymbol{U})\right) + \frac{\partial}{\partial \eta_2}\left(J\frac{\partial \eta_2}{\partial x}\boldsymbol{F}(\boldsymbol{U})\right) +$$

$$\frac{\partial}{\partial \eta_1}\left(J\frac{\partial \eta_1}{\partial y}\boldsymbol{G}(\boldsymbol{U})\right) + \frac{\partial}{\partial \eta_2}\left(J\frac{\partial \eta_2}{\partial y}\boldsymbol{G}(\boldsymbol{U})\right) = \boldsymbol{0} \tag{8}$$

又因为

$$J\frac{\partial \eta_1}{\partial t} = \frac{\partial y}{\partial \tau}\frac{\partial x}{\partial \eta_2} - \frac{\partial x}{\partial \tau}\frac{\partial y}{\partial \eta_2} \qquad J\frac{\partial \eta_2}{\partial t} = \frac{\partial x}{\partial \tau}\frac{\partial y}{\partial \eta_1} - \frac{\partial y}{\partial \tau}\frac{\partial x}{\partial \eta_1}$$

$$J\frac{\partial \eta_1}{\partial x} = \frac{\partial y}{\partial \eta_2} \qquad J\frac{\partial \eta_2}{\partial x} = -\frac{\partial y}{\partial \eta_1} \qquad J\frac{\partial \eta_1}{\partial y} = -\frac{\partial x}{\partial \eta_2} \qquad J\frac{\partial \eta_2}{\partial y} = \frac{\partial x}{\partial \eta_1}$$

于是可得到如下二维曲线坐标系方程的守恒型形式:

$$\frac{\partial}{\partial \tau}(J\boldsymbol{U}) + \frac{\partial}{\partial \eta_1}\left((y_\tau x_{\eta_2} - x_\tau y_{\eta_2})\boldsymbol{U} + y_{\eta_2}\boldsymbol{F}(\boldsymbol{U}) - x_{\eta_2}\boldsymbol{G}(\boldsymbol{U})\right) +$$

$$\frac{\partial}{\partial \eta_2}\left((x_\tau y_{\eta_1} - y_\tau x_{\eta_1})\boldsymbol{U} - y_{\eta_1}\boldsymbol{F}(\boldsymbol{U}) + x_{\eta_1}\boldsymbol{G}(\boldsymbol{U})\right) = \boldsymbol{0}$$

2　边界条件的处理

考虑到问题的复杂性,仅以二维问题为例说明处理技巧.边界条件一般分固壁边界条件、来流条件和下游条件等.这里为了在方形计算域中的方形网格下数值求解浅水波方程的需要,探讨了物理域中的固壁边界条件经过全局同胚自适应网格生成变换后在计算域中的形式.设 Γ 是物理域固壁边界线(如图 1),P 是 Γ 上的任一点,$\boldsymbol{\tau}=(-l_2,l_1)$ 和 $\boldsymbol{n}=(l_1,l_2)$ 分别是 Γ 上过 P 点的单位切向量和单位法向量.图 2 是 P 点处物理域固壁边界 Γ 的局部示意图,A' 是流场中点 A 关于固壁的对称点.设 u,v 是沿直角坐标 x 轴和 y 轴方向的水流速度,z 是 P 点处的水深,$u_{/\!/}$ 和 $u_{\perp}$ 分别表示沿切向 $\boldsymbol{\tau}$ 和法向 $\boldsymbol{n}$ 方向的水流速度.

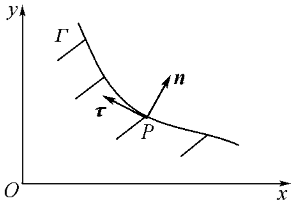

图 1　物理域中的固壁边界示意图

Fig.1　Wall boundary in physical domain

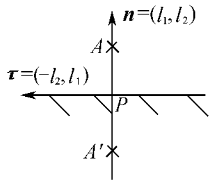

图 2　物理域中的固壁边界局部示意图

Fig.2　Local wall boundary in physical domain

一般物理域不渗透固壁边界条件可以用下面的形式给出:

$$u_{\perp}(P)=0 \tag{9}$$

$$u_{\perp}(A)=-u_{\perp}(A') \tag{10}$$

$$u_{/\!/}(A)=u_{/\!/}(A') \tag{11}$$

$$z(A)=z(A') \tag{12}$$

把 $u_{\perp}(A)$,$u_{\perp}(A')$,$u_{/\!/}(A)$,$u_{/\!/}(A')$ 和 $z(A)$,$z(A')$ 在点 P 处,沿法向 $\boldsymbol{n}$ 展开,代入式(10),(11),(12)可推得

$$\left.\frac{\partial^2 u_{\perp}}{\partial \boldsymbol{n}^2}\right|_P=0 \qquad \left.\frac{\partial u_{/\!/}}{\partial \boldsymbol{n}}\right|_P=0 \tag{13}$$

$$\left.\frac{\partial z}{\partial \boldsymbol{n}}\right|_P=0 \tag{14}$$

由于
$$u_{\perp}(A)=u(A)l_1+v(A)l_2 \qquad u_{/\!/}(A)=-u(A)l_2+v(A)l_1 \tag{15}$$

从而式(13) 和(14) 可写为

$$\left.\left(\frac{\partial^2 u(A)}{\partial \boldsymbol{n}^2}l_1+\frac{\partial^2 v(A)}{\partial \boldsymbol{n}^2}l_2\right)\right|_P=0 \tag{16}$$

$$\left.\left(-\frac{\partial u(A)}{\partial \boldsymbol{n}}l_2+\frac{\partial v(A)}{\partial \boldsymbol{n}}l_1\right)\right|_P=0 \tag{17}$$

进一步可得到式(15),(16),(17)在直角坐标系(x,y)下的微分方程形式:

$$\left.(z_x l_1+z_y l_2)\right|_P=0 \tag{18}$$

$$\left.\left[l_1(l_1^2 u_{xx}+2l_1 l_2 u_{xy}+l_2^2 u_{yy})+l_2(l_1^2 v_{xx}+2l_1 l_2 v_{xy}+l_2^2 v_{yy})\right]\right|_P=0 \tag{19}$$

$$\left.\left[-l_2(l_1 u_x+l_2 u_y)+l_1(l_1 v_x+l_2 v_y)\right]\right|_P=0 \tag{20}$$

在曲线坐标网格生成变换下,式(18)~(20)可化为方形计算域中固壁边界条件的约束关系式:

$$\left.\left[l_1(y_{\eta_2}z_{\eta_1}-y_{\eta_1}z_{\eta_2})+l_2(x_{\eta_1}z_{\eta_2}-x_{\eta_2}z_{\eta_1})\right]\right|_P=0 \tag{21}$$

$$\left.\left[b_1 u_{\eta_1\eta_1}+b_2 u_{\eta_1\eta_2}+b_3 u_{\eta_2\eta_2}+b_4 u_{\eta_1}+b_5 u_{\eta_2}+c_1 v_{\eta_1\eta_1}+c_2 v_{\eta_1\eta_2}+c_3 v_{\eta_2\eta_2}+c_4 v_{\eta_1}+c_5 v_{\eta_2}\right]\right|_P=0 \tag{22}$$

$$\left.\left[a_1(l_1 v_{\eta_1}-l_2 u_{\eta_1})+a_2(l_1 v_{\eta_2}-l_2 u_{\eta_2})\right]\right|_P=0 \tag{23}$$

其中

$$a_1 = l_1 y_{\eta_2} - l_2 x_{\eta_2} \qquad a_2 = - l_1 y_{\eta_1} + l_2 x_{\eta_1}$$

$$b_1 = l_1 a_1^2 \qquad b_2 = 2 l_1 a_1 a_2 \qquad b_3 = l_1 a_2^2$$

$$b_4 = l_1 \{\frac{1}{J}[- a_1^2 J_{\eta_1} + J_{\eta_2}(y_{\eta_1} y_{\eta_2} - x_{\eta_1} y_{\eta_2} + x_{\eta_1} x_{\eta_2})] + l_1^2(y_{\eta_2} y_{\eta_1 \eta_2} - y_{\eta_1} y_{\eta_2 \eta_2}) + 2 l_1 l_2 (x_{\eta_1} y_{\eta_2 \eta_2} - x_{\eta_2} y_{\eta_1 \eta_1}) + l_2^2 (x_{\eta_2} x_{\eta_1 \eta_2} - x_{\eta_1} x_{\eta_2 \eta_2})\}$$

$$b_5 = l_1 \{\frac{1}{J}[- a_2^2 J_{\eta_2} + J_{\eta_1}(y_{\eta_1} y_{\eta_2} - x_{\eta_2} y_{\eta_1} + x_{\eta_1} x_{\eta_2})] + l_1^2(y_{\eta_1} y_{\eta_1 \eta_2} - y_{\eta_2} y_{\eta_1 \eta_1}) + 2 l_1 l_2 (x_{\eta_2} y_{\eta_1 \eta_1} - x_{\eta_1} y_{\eta_1 \eta_2}) + l_2^2 (x_{\eta_1} x_{\eta_1 \eta_2} - x_{\eta_2} x_{\eta_1 \eta_1})\}$$

$$c_1 = l_2 a_1^2 \qquad c_2 = 2 l_2 a_1 a_2 \qquad c_3 = l_2 a_2^2$$

$$c_4 = l_2 \{\frac{1}{J}[- a_1^2 J_{\eta_1} + J_{\eta_2}(y_{\eta_1} y_{\eta_2} - x_{\eta_1} y_{\eta_2} + x_{\eta_1} x_{\eta_2})] + l_1^2(y_{\eta_2} y_{\eta_1 \eta_2} - y_{\eta_1} y_{\eta_2 \eta_2}) + 2 l_1 l_2 (x_{\eta_1} y_{\eta_2 \eta_2} - x_{\eta_2} y_{\eta_1 \eta_1}) + l_2^2 (x_{\eta_2} x_{\eta_1 \eta_2} - x_{\eta_1} x_{\eta_2 \eta_2})\}$$

$$c_5 = l_2 \{\frac{1}{J}[- a_2^2 J_{\eta_2} + J_{\eta_1}(y_{\eta_1} y_{\eta_2} - x_{\eta_2} y_{\eta_1} + x_{\eta_1} x_{\eta_2})] + l_1^2(y_{\eta_1} y_{\eta_1 \eta_2} - y_{\eta_2} y_{\eta_1 \eta_1}) + 2 l_1 l_2 (x_{\eta_2} y_{\eta_1 \eta_1} - x_{\eta_1} y_{\eta_1 \eta_2}) + l_2^2 (x_{\eta_1} x_{\eta_1 \eta_2} - x_{\eta_2} x_{\eta_1 \eta_1})\}$$

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Notes of numerical computing technology with curvilinear coordinate grids

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Abstract: Based on the former research of keeping variables of conservation characteristics in the conservation form, a general treatment technique is given to keep the conservation law in the conservation form. Besides, a technique is proposed for transforming impervious wall boundaries in the physical domain into corresponding forms in the computing domain by curvilinear coordinate generation to ensure the feasibility of computation.

Key words: conservation law; curvilinear coordinate grids; boundary condition